**Stefan problems and the Penrose–Fife phase field model**

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Abstract.

This paper is concerned with singular Stefan problems in which the heat flux is proportional to the gradient of the inverse absolute temperature. Both the standard interphase equilibrium conditions and phase relaxations are considered. These problems turn out to be the natural limiting cases of a thermodynamically consistent model for diffusive phase transitions proposed by Penrose and Fife. By supplying the systems of equations with suitable initial and boundary conditions, a rigorous asymptotic analysis is performed, and the unique solutions to the different Stefan problems are derived as asymptotic limits of the solutions to the Penrose–Fife phase–field problem.

## 1. Introduction

It is well–known that a weak formulation for the Stefan problem is based on the pair of equations

$$\partial_t(c_0\,\theta\,+\,L\,\chi)\,+\,\mathrm{div}\Big(\tilde{k}(\theta)\,\nabla\Big(\frac{1}{\theta}\Big)\Big)\;=\;g\quad\text{in } Q:=\Omega\,\times\,(0,T)\,, \tag{1.1}$$

$$\chi\,\in\,H\,(\theta\,-\,\theta_C)\quad\text{in } Q\,, \tag{1.2}$$

for the absolute temperature $\theta\,:\,Q\to\mathbb{R}$ and the phase variable $\chi\,:\,Q\to[0,1]$. Here, $\Omega\,\subseteq\,\mathbb{R}^3$ denotes a smooth bounded domain with boundary $\Gamma$, $T\,>\,0$ stands for a final time, and $\partial_t\,,\mathrm{div}\,,\nabla$ indicate time derivative, spatial divergence and gradient operators, respectively. The datum $g\,:\,Q\to\mathbb{R}$ gives the heat supply, the constants $c_0$ and $L$ are referred to as specific heat and latent heat, and $\tilde{k}\,:\,(0,+\infty)\to\mathbb{R}$ is a positive function depending on the thermal conductivity of the material. It is worth recalling that the variable $\chi$ usually represents the local concentration of one of the two phases, for instance of water in a water–ice system. Thus, with $\theta_C$ being the critical temperature of phase change and $H$ denoting the Heaviside graph, the inclusion (1.2) postulates that $\chi = 0$ where $\theta < \theta_C$ (solid region), $\chi = 1$ where $\theta > \theta_C$ (liquid region), and $\chi \in [0,1]$ where $\theta = \theta_C$ (mushy region).

The equations (1.1), (1.2) can be derived following the usual approach of thermodynamics (see [8]). Thus, (1.2) is a constitutive relation complying with the second principle, and (1.1) results from the balance of internal energy with the heat flux $\vec{q}$ given by

$$\vec{q} = \tilde{k}(\theta)\nabla\left(\frac{1}{\theta}\right). \tag{1.3}$$

The classical Stefan problem has been widely investigated (cf., e.g., [7] and the references therein) in the framework of the Fourier law which corresponds to the choice $\tilde{k}(\theta) = k\,\theta^2$ in (1.3), for some constant $k > 0$. On the contrary, this paper is characterized by the alternative assumption that $\tilde{k}$ is a constant,

$$\tilde{k}(\theta) = k > 0. \tag{1.4}$$

In fact, we study the system (1.1)–(1.2) and some perturbations thereof within the above setting. Let us note that (1.4) arises quite naturally as a first choice in (1.3) and has the advantage that the consequent heat flux law keeps the absolute temperature away from the singular value $\theta = 0$, as one expects from the physical point of view.

In this connection, we point out that very recently some effort has been directed towards the analysis (see [9-15, 20, 22]) of the phase–field model proposed by Penrose and Fife [18, 19] including the position (1.4). In the case when the order parameter $\chi$ is not conserved (for the other case we refer to [1, 2, 18]), a general version of the Penrose–Fife system reads

$$\partial_t\Big(c_0\,\theta - \lambda(\chi)\Big) + k\,\Delta\left(\frac{1}{\theta}\right) = g \quad \text{in } Q, \tag{1.5}$$

$$\delta\,\chi_t - \varepsilon\,\Delta\chi + \beta(\chi) \ni \sigma'(\chi) + \frac{\lambda'(\chi)}{\theta} \quad \text{in } Q, \tag{1.6}$$

with smooth functions $\lambda, \sigma$ and a maximal monotone graph $\beta$ from $\mathbb{R}$ to $\mathbb{R}$. Here, $\delta$ and $\varepsilon$ are small positive parameters governing the dissipation terms of (1.6). To realize that also (1.5)–(1.6) is thermodynamically consistent, the interested reader can find a rigorous justification in [2], where various phase transition models are studied. In particular, the standard phase–field model [5] can be recovered from (1.5)–(1.6) by suitably fixing $\beta, \sigma, \lambda$ and linearizing with respect to $\theta - \theta_C$ (see also [19]).

Let us return to (1.1)–(1.2). In view of (1.4), a comparison between (1.1) and (1.5) shows that the equation

$$\partial_t(c_0\,\theta + L\,\chi) + k\,\Delta\left(\frac{1}{\theta}\right) = g \quad \text{in } Q \tag{1.7}$$

can be regarded as a reduction of (1.5) to the simple situation $\lambda(\chi) = -L\,\chi$. Moreover, we may equivalently rewrite the law (1.2) as $H^{-1}(\chi) \ni \theta - \theta_C$ or (multiplying by

$L\,(\theta_C\,\theta)^{-1} > 0$) as

$$H^{-1}(\chi) \ni L\Big(\frac{1}{\theta_C} - \frac{1}{\theta}\Big) \quad \text{in } Q\,, \tag{1.8}$$

so that (1.8) corresponds to (1.6) for $\sigma(\chi) = L\chi/\theta_C$ and $\beta = H^{-1}$, provided that $\delta = \varepsilon = 0$.

Owing to this relationship, our idea was to study an initial–boundary value problem for the system (1.7)–(1.8) by approximating it with the analogous problem for $\delta > 0$, $\varepsilon > 0$, and then letting $\delta$ and $\varepsilon$ tend to zero. Such a procedure looks somewhat opportune. Indeed, one can use the smooth solutions already found for the general situation (1.5)–(1.6) in the works of Laurençot [13–15] and Kenmochi–Niezgódka [12], who extended techniques originally developed in [20] and [22]. In addition, the successive asymptotic analysis of the Penrose–Fife initial–boundary value problem seems to be, by itself, interesting and allows us to discuss the intermediate cases $\delta > 0$, $\varepsilon = 0$ and $\delta = 0$, $\varepsilon > 0$, which can be viewed as Stefan problems with just one form of dissipation. The former may be compared with the relaxed Stefan model considered in [21], and it has already been investigated in the paper [6] (but in a different framework, including the nonlinearities $\lambda, \sigma$ of (1.5)–(1.6), and with the aid of regularity results not exploited here).

In order to make the above statements more precise, let us first provide boundary and initial conditions to (1.7)–(1.8). We choose a boundary condition linear with respect to $1/\theta$, namely

$$k\,\frac{\partial}{\partial n}\Big(\frac{1}{\theta}\Big) = \gamma\Big(\frac{1}{\theta_\Gamma} - \frac{1}{\theta}\Big) \quad \text{in } \Sigma := \Gamma \times (0,T)\,, \tag{1.9}$$

where $\partial/\partial n$ denotes the outward normal derivative, $\gamma : \Sigma \to \mathbb{R}$ and $\theta_\Gamma : \Sigma \to \mathbb{R}$ are given positive functions. In particular, $\theta_\Gamma$ represents the outside temperature. Thus (1.9) asserts that the heat flux is proportional, by the factor $\gamma$, to the difference of the inverse absolute temperatures between the exterior and the interior of the body (for other possible right hand sides in (1.9), see [6, Section 5] and the later Remark 4.8). Next, letting $e_0 : \Omega \to \mathbb{R}$ measure the initial enthalpy, we prescribe that (cf. (1.7))

$$(c_0\,\theta + L\,\chi)(\,\cdot\,,0) = e_0 \quad \text{in } \Omega\,. \tag{1.10}$$

Besides the initial–boundary value problem (1.7)–(1.10), we also consider its two variations obtained by substituting (1.8) with either

$$\delta\,\chi_t + H^{-1}(\chi) \ni L\Big(\frac{1}{\theta_C} - \frac{1}{\theta}\Big) \quad \text{in } Q \tag{1.11}$$

or

$$-\,\varepsilon\,\Delta\chi + H^{-1}(\chi) \ni L\Big(\frac{1}{\theta_C} - \frac{1}{\theta}\Big) \quad \text{in } Q\,. \tag{1.12}$$

The formulations of the two additional problems have to be completed by setting either an initial condition or a boundary condition, respectively, for $\chi$. Therefore we add

$$\chi(\,\cdot\,,0) = \chi_0 \quad \text{in } \Omega \tag{1.13}$$

to (1.11) and, according to [18], we couple (1.12) with the no–flux condition

$$\frac{\partial \chi}{\partial n} = 0 \quad \text{in } \Sigma. \tag{1.14}$$

Summarizing, we are concerned with the three problems (1.7)–(1.10) (pure Stefan); (1.7), (1.9)–(1.11), (1.13) (Stefan relaxed in time); (1.7), (1.9)–(1.10), (1.12), (1.14) (Stefan relaxed in space). We approximate them by the following system of equations and conditions

$$\partial_t (c_0 \theta + L \chi) + k \Delta\Big(\frac{1}{\theta}\Big) = g_{\delta\varepsilon} \quad \text{in } Q, \tag{1.15}$$

$$\delta \chi_t - \varepsilon \Delta\chi + H^{-1}(\chi) \ni L \Big(\frac{1}{\theta_C} - \frac{1}{\theta}\Big) \quad \text{in } Q, \tag{1.16}$$

$$k \frac{\partial}{\partial n} \Big(\frac{1}{\theta}\Big) = \gamma \Big(\frac{1}{\theta_\Gamma} - \frac{1}{\theta}\Big), \quad \frac{\partial \chi}{\partial n} = 0 \quad \text{in } \Sigma, \tag{1.17}$$

$$(c_0 \theta + L \chi)(\,\cdot\,, 0) = e_{0\delta\varepsilon}, \quad \chi(\,\cdot\,, 0) = \chi_{0\delta\varepsilon}, \tag{1.18}$$

where $\{g_{\delta\varepsilon}\}, \{e_{0\delta\varepsilon}\}, \{\chi_{0\delta\varepsilon}\}$ are sequences of data with suitable smoothness and convergence properties. Obviously, one needs that $g_{\delta\varepsilon}, e_{0\delta\varepsilon}, \chi_{0\delta\varepsilon}$ approach $g, e_0, \chi_0$, respectively, as $\varepsilon$ or $\delta$ or both $\varepsilon$ and $\delta$ tend to 0.

For any $\delta > 0$, $\varepsilon > 0$, under suitable assumptions, the problem (1.15)–(1.18) has a unique solution $(\theta_{\delta\varepsilon}, \chi_{\delta\varepsilon})$ (cf. [12]). Then the asymptotic analysis can start. The crucial step consists in deriving global estimates, independent of $\delta$ and $\varepsilon$, for $\theta_{\delta\varepsilon}$ and $\chi_{\delta\varepsilon}$. Owing to these estimates, we will be able to pass to the limit in (1.15)–(1.16) by compactness and monotonicity arguments. We perform three limit procedures, letting first $\varepsilon \searrow 0$, then $\delta \searrow 0$, and finally both $\varepsilon$ and $\delta$, without any order relation between the two parameters, tend to $0$. We find that all weak–star limits $\theta, \chi$ of subsequences of $\{\theta_{\delta\varepsilon}\}, \{\chi_{\delta\varepsilon}\}$ must yield a weak solution to the following Stefan problem

- (1.7), (1.9)–(1.11), (1.13) in the first case ($\varepsilon \searrow 0$),
- (1.7), (1.9)–(1.10), (1.12), (1.14) in the second case ($\delta \searrow 0$),
- (1.7)–(1.10) in the third case ($\varepsilon \searrow 0$, $\delta \searrow 0$).

As a consequence of this analysis, we will establish three results of global existence. Moreover, since we can show that each of the three limit problems admits only one solution, the convergences $\theta_{\delta\varepsilon} \to \theta$, $\chi_{\delta\varepsilon} \to \chi$ hold for the whole sequences in any limit procedure. Concerning the uniqueness proof, we should point out that an essential role is played by the special form of the boundary condition in (1.9).

Precise formulations of the problems are provided in Section 2, along with statements of the main results, which will be proved in the subsequent sections. Section 3 contains the proof of the uniform estimates, Section 4 is devoted to the passages to the limit, and Section 5 brings the details of the uniqueness argument.

First, we fix some notation. Set $V := H^1(\Omega)$ and identify $H := L^2(\Omega)$ with its dual space $H'$, so that $V \subset H \subset V'$ with dense and compact injections. Let $(\cdot\,,\,\cdot)$ represent either the duality pairing between $V'$ and $V$ or the scalar product in $H$. The norms in both $L^2(\Omega)$ and $(L^2(\Omega))^3$ are simply denoted by $\|\cdot\|$, while $\|\cdot\|_\Gamma$ stands for the norm in $L^2(\Gamma)$. The trace of a function $v \in H^1(\Omega)$ on the boundary $\Gamma$ is indicated by $v_{|\Gamma} \in H^{1/2}(\Gamma)$ or, if no confusion can arise, just by $v$.

From now on, let $\beta$ coincide with the maximal monotone graph $H^{-1}$, namely

$$\beta(r) = \begin{cases} (-\infty\,,0] & \text{if} \quad r = 0 \\ \{0\} & \text{if} \quad 0 < r < 1 \\ [0\,,+\infty) & \text{if} \quad r = 1 \end{cases} . \tag{2.1}$$

Therefore, $\beta$ acts from [0, 1] to $\mathbb{R}$. Setting $u_C = 1/\theta_C$ and $\gamma/\theta_\Gamma = \zeta$ (cf. (1.8) and (1.9)), let us recall that $c_0\,, L\,, k\,, u_C$ are known positive constants and that $g\,,\gamma\,,\zeta\,,e_0\,,\chi_0$ are given functions defined on $Q\,,\Sigma\,,\Omega$, respectively.

For the sake of convenience, the Stefan problems outlined in (1.7)–(1.14) will be formulated in terms of four unknowns. Besides the absolute temperature $\theta$ and the phase density $\chi$, we make use of the auxiliary variables $u$ and $\xi$, related to $\theta$ and $\chi$ by the conditions $u = 1/\theta$ and $\xi \in \beta\,(\chi)$. However, before stating the variational formulations, we prescribe the common assumptions on the data. It is required that

$$g \in L^\infty(Q)\,, \tag{2.2}$$

$$\gamma \in L^\infty(\Sigma)\,, \qquad \gamma \geq c \quad \text{a.e. in } \Sigma\,, \qquad \gamma_t \in L^\infty(\Sigma)\,, \tag{2.3}$$

$$\zeta \in L^\infty(\Sigma)\,, \qquad \zeta \geq 0 \quad \text{a.e. in } \Sigma\,, \qquad \zeta_t \in L^\infty(\Sigma)\,, \tag{2.4}$$

$$e_0 = c_0\,\theta_0 + L\,\chi_0 \tag{2.5}$$

for some positive constant $c$ and for two initial values $\theta_0\,,\chi_0$ fulfilling

$$\theta_0 \in H^1(\Omega)\,, \qquad \theta_0 > 0 \quad \text{a.e. in } \Omega\,, \qquad \ln(\theta_0) \in L^\infty(\Omega)\,, \tag{2.6}$$

$$\chi_0 \in H^1(\Omega)\,, \qquad 0 \leq \chi_0 \leq 1 \quad \text{a.e. in } \Omega\,. \tag{2.7}$$

Note that (2.6) yields $\theta_0^{\,r} \in H^1(\Omega) \cap L^\infty(\Omega)$ for any $r \in \mathbb{R}$. In particular, there are two positive constants $a\,,b$ such that

$$u_0 := \frac{1}{\theta_0} \in H^1(\Omega)\,, \qquad a \leq u_0 \leq b \quad \text{a.e. in } \Omega\,. \tag{2.8}$$

Then, letting $\delta > 0$ and $\varepsilon > 0$, we can define precisely the three singular Stefan problems we deal with in this paper.

**Problem (P$_\delta$).** Find a quadruple $(\,\theta\,,u\,,\chi\,,\xi\,)$ satisfying

$$\theta \in L^\infty(0\,,T;L^2(\Omega))\,, \quad u \in L^\infty(0\,,T;H^1(\Omega))\,, \tag{2.9}$$

$$\chi \in L^\infty(Q), \quad \xi \in L^\infty(0,T;L^2(\Omega)) \tag{2.10}$$

$$\theta > 0, \quad u = \frac{1}{\theta} \quad \text{a.e. in } Q, \tag{2.11}$$

$$0 \le \chi \le 1, \quad \xi \in \beta(\chi) \quad \text{a.e. in } Q, \tag{2.12}$$

$$c_0\,\theta + L\chi \in W^{1,\infty}(0,T;V'), \tag{2.13}$$

$$\begin{aligned}\Big(\partial_t(c_0\,\theta + L\chi)(\cdot,t), v\Big) &= k\int_\Omega \nabla u(\cdot,t)\cdot\nabla v + \int_\Gamma (\gamma u - \zeta)(\cdot,t)\,v \\ &+ \Big(g(\cdot,t), v\Big) \qquad \forall\, v \in V, \quad \text{for a.e. } t \in (0,T),\end{aligned} \tag{2.14}$$

$$(c_0\,\theta + L\chi)(\cdot,0) = e_0 \quad \text{in } V', \tag{2.15}$$

and such that

$$\chi \in W^{1,\infty}(0,T;L^2(\Omega)), \tag{2.16}$$

$$\delta\,\chi_t + \xi = L(u_C - u) \quad \text{a.e. in } Q, \tag{2.17}$$

$$\chi(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{2.18}$$

**Problem (P$_\varepsilon$).** Find a quadruple $(\theta, u, \chi, \xi)$ satisfying (2.9)–(2.15) and

$$\chi \in L^\infty(0,T;H^2(\Omega)), \tag{2.19}$$

$$-\varepsilon\,\Delta\chi + \xi = L(u_C - u) \quad \text{a.e. in } Q, \tag{2.20}$$

$$\frac{\partial\chi}{\partial n} = 0 \quad \text{a.e. in } \Sigma. \tag{2.21}$$

**Problem (P).** Find $(\theta, u, \chi, \xi)$ satisfying (2.9)–(2.15) and

$$\xi = L(u_C - u) \quad \text{a.e. in } Q. \tag{2.22}$$

**Remark 2.1.** Observe that (2.14) provides a weak formulation of (1.7) coupled with the boundary condition (1.9) (where $\gamma/\theta_\Gamma = \zeta$). The initial condition (2.15) makes sense even in the space $L^2(\Omega)$ (and consequently a.e. in $\Omega$): in fact, due to (2.9)–(2.10) and (2.13), $c_0\,\theta + L\chi$ is a weakly continuous function from $[0,T]$ into $L^2(\Omega)$. Regarding (2.12), we notice that the statement $0 \le \chi \le 1$ could be omitted since this information is already contained in the inclusion $\xi \in \beta(\chi)$ (cf. (2.1)). Also (2.11) can be presented in terms of maximal monotone operators, as done in [12]. Indeed, it suffices to introduce the maximal monotone graph

$$\rho(r) = -\frac{1}{r}, \qquad 0 < r < +\infty, \tag{2.23}$$

and to set $-u \in \rho(\theta)$ a.e. in $Q$.

For each one of the problems we have an existence and uniqueness result.

**Theorem 2.2.** *Under the assumptions (2.1)–(2.8), there exists one and only one solution $(\theta, u, \chi, \xi)$ of Problem ($\boldsymbol{P}_\delta$). Moreover, $u$ and $\chi$ fulfil*

$$u \in H^1(0,T;\, L^{3/2}(\Omega)), \tag{2.24}$$

$$\chi \in L^\infty(0,T;\, H^1(\Omega)). \tag{2.25}$$

**Theorem 2.3.** *Assume that (2.1)–(2.8) and*

$$\chi_0 \in H^2(\Omega), \quad \frac{\partial \chi_0}{\partial n} = 0 \quad a.e.\ in\ \Gamma, \tag{2.26}$$

$$-\varepsilon\, \Delta\chi_0 + \beta(\chi_0) \ni L\,(u_C - u_0) \quad a.e.\ in\ \Omega \tag{2.27}$$

*hold. Then Problem ($\boldsymbol{P}_\varepsilon$) admits a unique solution $(\theta, u, \chi, \xi)$ satisfying (2.24) and*

$$\chi \in H^1(0,T;\, H^1(\Omega)). \tag{2.28}$$

**Theorem 2.4.** *Assume that (2.1)–(2.8) and*

$$\beta(\chi_0) \ni L\,(u_C - u_0) \quad a.e.\ in\ \Omega \tag{2.29}$$

*hold. Then Problem ($\boldsymbol{P}$) has one and only one solution $(\theta, u, \chi, \xi)$ fulfilling (2.24).*

**Remark 2.5.** The additional assumptions (2.26)–(2.27) and (2.29) force the initial values $\chi_0, u_0$ (see (2.5)–(2.8)) to be suitably compatible in the problems where the phase relationship takes a stationary form (compare (2.20) and (2.22) with (2.17)). However, let us emphasize the space and time smoothness properties (2.16), (2.25) and (2.19), (2.28) of the phase variable $\chi$ in the relaxed problems ($\mathbf{P}_\delta$) and ($\mathbf{P}_\varepsilon$), respectively. In particular, (2.19) and (2.28) ensure that $\chi \in C^0(\overline{Q})$ for the solution to Problem ($\mathbf{P}_\varepsilon$).

Next, we consider the approximating system (1.15)–(1.18). Let the sequences $g_{\delta\varepsilon}, e_{0\delta\varepsilon}, \chi_{0\delta\varepsilon}$ satisfy

$$g_{\delta\varepsilon},\ \partial_t g_{\delta\varepsilon} \in L^\infty(Q), \tag{2.30}$$

$$e_{0\delta\varepsilon} = c_0\,\theta_{0\delta\varepsilon} + L\,\chi_{0\delta\varepsilon}, \tag{2.31}$$

$$\theta_{0\delta\varepsilon} \in H^1(\Omega) \cap L^\infty(\Omega), \qquad \theta_{0\delta\varepsilon} > 0 \quad \text{a.e. in } \Omega, \tag{2.32}$$

$$u_{0\delta\varepsilon} := \frac{1}{\theta_{0\delta\varepsilon}} \in H^1(\Omega) \cap L^\infty(\Omega), \tag{2.33}$$

$$\chi_{0\delta\varepsilon} \in H^2(\Omega), \qquad \frac{\partial \chi_{0\delta\varepsilon}}{\partial n} = 0 \quad \text{a.e. in } \Gamma, \tag{2.34}$$

$$0 \le \chi_{0\delta\varepsilon} \le 1 \quad \text{a.e. in } \Omega \tag{2.35}$$

for all $\delta > 0, \varepsilon > 0$. Combining the results of [12] with those of [13, 14], it is not difficult to establish the following existence and uniqueness theorem.

**Proposition 2.6.** *Under the assumptions (2.3)–(2.4), (2.30)–(2.35) there is one and only one quadruple* $(\theta_{\delta\varepsilon}, u_{\delta\varepsilon}, \chi_{\delta\varepsilon}, \xi_{\delta\varepsilon})$ *fulfilling*

$$\theta_{\delta\varepsilon} \in H^1(0,T;\, L^2(\Omega)) \cap L^\infty(0,T;\, H^1(\Omega)) \cap L^\infty(Q), \tag{2.36}$$

$$u_{\delta\varepsilon} \in H^1(0,T;\, L^2(\Omega)) \cap L^2(0,T;\, H^2(\Omega)) \cap L^\infty(Q), \tag{2.37}$$

$$\chi_{\delta\varepsilon} \in W^{1,\infty}(0,T; L^2(\Omega)) \cap H^1(0,T; H^1(\Omega)) \cap L^\infty(0,T;\, H^2(\Omega)), \tag{2.38}$$

$$\xi_{\delta\varepsilon} \in L^\infty(0,T;\, L^2(\Omega)), \tag{2.39}$$

$$\theta_{\delta\varepsilon} > 0, \qquad u_{\delta\varepsilon} = \frac{1}{\theta_{\delta\varepsilon}} \quad a.e.\ in\ Q, \tag{2.40}$$

$$0 \le \chi_{\delta\varepsilon} \le 1, \qquad \xi_{\delta\varepsilon} \in \beta(\chi_{\delta\varepsilon}) \quad a.e.\ in\ Q, \tag{2.41}$$

$$\partial_t (c_0 \theta_{\delta\varepsilon} + L\chi_{\delta\varepsilon}) + k\Delta u_{\delta\varepsilon} = g_{\delta\varepsilon} \quad a.e.\ in\ Q, \tag{2.42}$$

$$\delta\, \partial_t \chi_{\delta\varepsilon} - \varepsilon\Delta\chi_{\delta\varepsilon} + \xi_{\delta\varepsilon} = L(u_C - u_{\delta\varepsilon}) \quad a.e.\ in\ Q, \tag{2.43}$$

$$k\frac{\partial u_{\delta\varepsilon}}{\partial n} + \gamma u_{\delta\varepsilon} = \zeta, \quad \frac{\partial \chi_{\delta\varepsilon}}{\partial n} = 0 \quad a.e.\ in\ \Sigma, \tag{2.44}$$

$$\theta_{\delta\varepsilon}(\,\cdot\,,0) = \theta_{0\delta\varepsilon}, \quad \chi_{\delta\varepsilon}(\,\cdot\,,0) = \chi_{0\delta\varepsilon} \quad a.e.\ in\ \Omega \tag{2.45}$$

*for any* $\delta > 0$ *and any* $\varepsilon > 0$.

**Remark 2.7.** Note that (2.42) and (2.44) allow us to deduce the variational equality (cf. (2.14))

$$\begin{aligned} \Big(\partial_t (c_0\theta_{\delta\varepsilon} + L\chi_{\delta\varepsilon})(\,\cdot\,,t), v\Big) &= k\int_\Omega \nabla u_{\delta\varepsilon}(\,\cdot\,,t)\cdot\nabla v \\ &\quad + \int_\Gamma (\gamma u_{\delta\varepsilon} - \zeta)(\,\cdot\,,t)v + \Big(g_{\delta\varepsilon}(\,\cdot\,,t), v\Big) \\ &\qquad \forall\, v \in V, \quad \text{for a.e. } t \in (0,T), \end{aligned} \tag{2.46}$$

whence it is easy to verify the regularity (2.13) for $c_0\theta_{\delta\varepsilon} + L\chi_{\delta\varepsilon}$. Besides, (2.45) and (2.31) entail the initial condition analogous to (2.15). Actually, owing to (2.31)–(2.33) one could equivalently prescribe initial values of $u_{\delta\varepsilon}$ and $\chi_{\delta\varepsilon}$ in place of (1.18) or (2.45).

Henceforth the problem (2.36)–(2.45) will be obviously named $(\mathbf{P}_{\delta\varepsilon})$. By investigating the asymptotic behaviour of $(\mathbf{P}_{\delta\varepsilon})$ as one or both of the parameters $\varepsilon$ and $\delta$ tend to zero, we can show that subsequences of the approximating solutions $(\theta_{\delta\varepsilon}, u_{\delta\varepsilon}, \chi_{\delta\varepsilon}, \xi_{\delta\varepsilon})$ converge (in the sense specified below) to solutions $(\theta, u, \chi, \xi)$ of the problems $(\mathbf{P}_{\delta})$, $(\mathbf{P}_{\varepsilon})$, $(\mathbf{P})$, thus proving the existence parts of Theorems 2.2–2.4. Moreover, because of uniqueness, the whole sequences will converge. In order to carry out the asymptotic analysis, we need, of course, that the approximating data $g_{\delta\varepsilon}, \theta_{0\delta\varepsilon}, \chi_{0\delta\varepsilon}$ satisfy some boundedness and convergence properties in addition to (2.30)–(2.35). Instead of detailing our requirements here, we prefer to select appropriate sequences of data and afterwards check them and infer the wanted conditions.

Therefore, in all the arguments we take

$$g_{\delta\varepsilon}(x, t) = \frac{1}{\delta\varepsilon}\int_0^t e^{-(t-\tau)/(\delta\varepsilon)} g(x, \tau) d\tau, \quad (x, t) \in Q, \tag{2.47}$$

while the other choices are expressed in the following statements.

**Theorem 2.8.** *Assume that (2.1)–(2.8), (2.30)–(2.35), (2.47), and*

$$\theta_{0\delta\varepsilon} = \theta_0, \tag{2.48}$$

$$\chi_{0\delta\varepsilon} - \varepsilon\Delta\chi_{0\delta\varepsilon} = \chi_0 \quad \text{a.e. in } \Omega \tag{2.49}$$

*hold. Let $(\theta, u, \chi, \xi)$ and $(\theta_{\delta\varepsilon}, u_{\delta\varepsilon}, \chi_{\delta\varepsilon}, \xi_{\delta\varepsilon})$ be the solutions to the problems $(\boldsymbol{P}_{\delta})$ and $(\boldsymbol{P}_{\delta\varepsilon})$, respectively. Then, as $\varepsilon$ tends to $0$, we have*

$$\theta_{\delta\varepsilon} \to \theta, \quad \xi_{\delta\varepsilon} \to \xi \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega)), \tag{2.50}$$

$$u_{\delta\varepsilon} \to u \quad \text{weakly star in } H^1(0, T; L^{3/2}(\Omega)) \cap L^\infty(0, T; H^1(\Omega)), \tag{2.51}$$

$$\chi_{\delta\varepsilon} \to \chi \quad \text{weakly star in } W^{1,\infty}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)). \tag{2.52}$$

**Theorem 2.9.** *Assume that (2.1)–(2.8), (2.26)–(2.27), (2.30)–(2.35), (2.47)–(2.48), and*

$$\chi_{0\delta\varepsilon} = \chi_0 \tag{2.53}$$

*hold. Let $(\theta, u, \chi, \xi)$ and $(\theta_{\delta\varepsilon}, u_{\delta\varepsilon}, \chi_{\delta\varepsilon}, \xi_{\delta\varepsilon})$ be the solutions to the problems $(\boldsymbol{P}_{\varepsilon})$ and $(\boldsymbol{P}_{\delta\varepsilon})$, respectively. Then, as $\delta$ tends to 0, we have the convergences (2.50), (2.51), and*

$$\chi_{\delta\varepsilon} \to \chi \quad \text{weakly star in } H^1(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)). \tag{2.54}$$

**Theorem 2.10.** *Assume that (2.1)–(2.8), (2.29)–(2.35), (2.47), and*

$$u_{0\delta\varepsilon} - \frac{a}{2}\chi_{0\delta\varepsilon} = u_0 - \frac{a}{2}\chi_0, \tag{2.55}$$

$$-\varepsilon\Delta\chi_{0\delta\varepsilon} + \beta(\chi_{0\delta\varepsilon}) \ni L(u_C - u_{0\delta\varepsilon}) \quad \text{a.e. in } \Omega \tag{2.56}$$

*hold (the constant $a$ being defined in (2.8)). Let $(\theta, u, \chi, \xi)$ and $(\theta_{\delta\varepsilon}, u_{\delta\varepsilon}, \chi_{\delta\varepsilon}, \xi_{\delta\varepsilon})$ denote the solutions to the problems* **(P)** *and* $(\mathbf{P}_{\delta\varepsilon})$*, respectively. Then, as $\varepsilon$ and $\delta$ go to 0, we have the convergences (2.50), (2.51), and*

$$\chi_{\delta\varepsilon} \to \chi \quad \textit{weakly star in } L^\infty(Q). \tag{2.57}$$

**Remark 2.11.** Thanks to (2.41), (2.57) is also fulfilled in the previous two cases. As far as problem **(P)** is concerned, one could wonder whether the assumption $\chi_0 \in H^1(\Omega)$ is really necessary to get the thesis of Theorems 2.4 and 2.10. In fact, (2.7), coupled with (2.8) and (2.29) (which seem to be essential for the outcome), prevents the interphase set $\{x \in \Omega : u_0(x) = u_C\}$ to be a smooth bi-dimensional surface. Our belief is that $\chi_0 \in H^1(\Omega)$ is rather a technical condition (at least for the conclusion of Theorem 2.4), whereas $\chi_0$ should be allowed to jump. For this matter and other possible generalizations of the results, we refer the reader to the remarks of Section 4.

## 3. Uniform estimates

In this section we derive estimates, independent of $\delta$ and $\varepsilon$, for the solutions $(\theta_{\delta\varepsilon}, u_{\delta\varepsilon}, \chi_{\delta\varepsilon}, \xi_{\delta\varepsilon})$ to the problem $(\mathbf{P}_{\delta\varepsilon})$ defined by Proposition 2.6. More precisely, our estimates may depend on $\delta$ (resp. $\varepsilon$) if such a parameter is fixed, like in Theorem 2.8 (resp. Theorem 2.9), but then and there we make distinctions. Anyway, in the sequel let $\bar\delta$ and $\bar\varepsilon$ represent two positive upper bounds for $\delta$ and $\varepsilon$,

$$0 < \delta \le \bar\delta, \quad 0 < \varepsilon \le \bar\varepsilon, \tag{3.1}$$

and let $C_i$, $i \in \mathbb{N}$, denote uniform constants not varying with $\delta$ or $\varepsilon$.

We start by pointing out some useful properties of the sequences approximating the data.

**Lemma 3.1.** *The functions $g_{\delta\varepsilon}$ introduced in (2.47) satisfy (2.30) and*

$$\|g_{\delta\varepsilon}\|_{L^\infty(Q)} \le C_1, \tag{3.2}$$

$$g_{\delta\varepsilon} \to g \quad \textit{strongly in } L^2(Q) \textit{ as } \varepsilon \searrow 0 \textit{ or } \delta \searrow 0. \tag{3.3}$$

**Proof.** This is quite elementary. Recalling (2.2), we just note that, for instance, $C_1 = \|g\|_{L^\infty(Q)}$. □

**Lemma 3.2.** *For any $\delta > 0$ and any $\varepsilon > 0$ the initial values $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}, \chi_{0\delta\varepsilon}, e_{0\delta\varepsilon}$ considered in the statements of Theorems 2.8–2.10 are uniquely determined and satisfy (2.31)–(2.35). Moreover, if $\xi_{0\delta\varepsilon} \in L^2(\Omega)$,*

$$\xi_{0\delta\varepsilon} \in \beta(\chi_{0\delta\varepsilon}) \quad \textit{a.e. in } \Omega, \tag{3.4}$$

*is defined by* $\xi_{0\delta\varepsilon} = 0$ *if* $(\boldsymbol{P}_\delta)$ *is concerned (see Theorem 2.8) and by* $\xi_{0\delta\varepsilon} = \varepsilon\,\Delta\chi_{0\delta\varepsilon} + L(u_C - u_{0\delta\varepsilon})$ *otherwise (see Theorem 2.9–2.10), then it holds*

$$\|u_{0\delta\varepsilon}\|_{H^1(\Omega)} \leq C_2\,, \quad \frac{a}{2} \leq u_{0\delta\varepsilon} \leq b + \frac{a}{2} \quad a.e.\ in\ \Omega\,, \tag{3.5}$$

$$(\delta + \varepsilon)\,\|\chi_{0\delta\varepsilon}\|^2_{H^1(\Omega)} \leq C_2\,, \tag{3.6}$$

$$\frac{1}{\delta}\,\|\varepsilon\,\Delta\chi_{0\delta\varepsilon} - \xi_{0\delta\varepsilon} + L(u_C - u_{0\delta\varepsilon})\|^2 \leq C_3\,, \tag{3.7}$$

$$e_{0\delta\varepsilon} \to e_0 \quad weakly\ in\ L^2(\Omega)\ as\ \varepsilon \searrow 0\ or\ \delta \searrow 0\,, \tag{3.8}$$

*where the constants* $a\,,b$ *are specified in (2.8) and* $C_3$ *depends on* $1/\delta$ *if* $(\boldsymbol{P}_\delta)$ *is intended as limit problem. In this case it is also required that*

$$\chi_{0\delta\varepsilon} \to \chi_0 \quad strongly\ in\ L^2(\Omega)\ as\ \varepsilon \searrow 0\,. \tag{3.9}$$

**Proof.** We first examine the situation in Theorem 2.8. Due to (2.48), (2.6), and (2.8), $\theta_{0\delta\varepsilon}$ and $u_{0\delta\varepsilon}$ fulfil (2.32)–(2.33) and (3.5). In view of (2.49) and (2.34), it turns out that $\chi_{0\delta\varepsilon}$ is the only solution of the elliptic variational equality

$$(\chi_{0\delta\varepsilon}\,,v) + \varepsilon\int_\Omega \nabla\chi_{0\delta\varepsilon}\cdot\nabla v = (\chi_0\,,v) \qquad \forall\ v \in H^1(\Omega)\,.$$

Since $\chi_0$ attains values between 0 and 1 (cf. (2.7)), a standard maximum principle argument enables us to deduce (2.35). Taking $v = \chi_{0\delta\varepsilon}$ above and comparing the terms in (2.49), we easily obtain

$$\varepsilon\,\|\nabla\chi_{0\delta\varepsilon}\|^2 + \|\varepsilon\,\Delta\chi_{0\delta\varepsilon}\|^2 \leq 2\,|\Omega|\,, \tag{3.10}$$

with $|\Omega|$ denoting the Lebesgue measure of the domain $\Omega$. Hence, it is straightforward to recover (3.7). The convergence (3.9) (which implies (3.8) because of (2.48), (2.31), and (2.5)) can be inferred via singular perturbation techniques (see [17]). As $\delta$ is fixed, to get (3.6) we must exploit the further condition $\chi_0 \in H^1(\Omega)$. Multiplying (2.49) by $-\Delta\chi_{0\delta\varepsilon}$ and integrating by parts, it results that

$$\frac{1}{2}\,\|\nabla\chi_{0\delta\varepsilon}\|^2 + \varepsilon\,\|\Delta\chi_{0\delta\varepsilon}\|^2 \leq \frac{1}{2}\,\|\nabla\chi_0\|^2 \tag{3.11}$$

and consequently (3.6) follows from (2.7). Next, let us consider the frameworks of Theorems 2.9 and 2.10. Note that (cf. (2.48), (2.53), (2.26)–(2.27)) in both cases $u_{0\delta\varepsilon}$ and $\chi_{0\delta\varepsilon}$ solve the system (2.55)–(2.56) supplied with (2.34). Owing to the suitable definition of $\xi_{0\delta\varepsilon}$, (3.7) is certainly fulfilled. On the other hand, besides showing (3.5)–(3.6) and (3.8), we have to check that there is a unique pair $(u_{0\delta\varepsilon}\,,\chi_{0\delta\varepsilon})$ satisfying (2.34), (2.55)–(2.56). To this end, it suffices to prove that the nonlinear elliptic problem (2.34),

$$\frac{L\,a}{2}\,\chi_{0\delta\varepsilon} - \varepsilon\,\Delta\chi_{0\delta\varepsilon} + \beta\,(\chi_{0\delta\varepsilon}) \ni L\,(u_C - u_0 + \frac{a}{2}\,\chi_0) \quad \text{a.e. in } \Omega \tag{3.12}$$

admits one and only one solution $\chi_{0\delta\varepsilon}$, finding $u_{0\delta\varepsilon}$ subsequently from (2.55). The uniqueness of $\chi_{0\delta\varepsilon}$ is entailed by the monotonicity of $\beta$ and can be verified by contradiction. The existence proof is based on standard methods of the theory of maximal monotone operators (see, e.g., [4] or [3]). Thus, one replaces in (3.12) the graph $\beta$ by its Yosida approximation,

$$\beta_m(r) = \begin{cases} m\,r & \text{if} \quad r < 0 \\ 0 & \text{if} \quad 0 \le r \le 1, \quad m \in \mathbb{N}, \\ m\,(r-1) & \text{if} \quad r > 1 \end{cases} \tag{3.13}$$

and denotes by $\chi_{0m}$ the solution to

$$\frac{L\,a}{2}\chi_{0m} - \varepsilon\,\Delta\chi_{0m} + \beta_m(\chi_{0m}) = L\,(u_C - u_0 + \frac{a}{2}\chi_0) \quad \text{a.e. in } \Omega\,, \tag{3.14}$$

subjected to the conditions in (2.34). In order to derive uniform estimates, we multiply (3.14) by $\chi_{0m} - \Delta\chi_{0m}$ and integrate by parts. Observe that

$$\int_\Omega \beta_m(\chi_{0m})\,(\chi_{0m} - \Delta\chi_{0m}) = \int_\Omega \left(\beta_m(\chi_{0m})\chi_{0m} + \beta'(\chi_{0m})\,|\nabla\chi_{0m}|^2\right) \ge 0$$

because of (3.13). As the right hand side of (3.14) belongs to $H^1(\Omega)$ (cf. (2.7)–(2.8)), applications of the elementary Young inequality allow us to conclude that

$$\frac{L\,a}{4}\,\|\chi_{0m}\|^2_{H^1(\Omega)} + \varepsilon\,\|\nabla\chi_{0m}\|^2 + \varepsilon\,\|\Delta\chi_{0m}\|^2$$
$$\le \frac{L}{a}\,\|u_C - u_0 + \frac{a}{2}\chi_0\|^2_{H^1(\Omega)}\,.$$

Then, setting $\xi_{0m} = \beta_m(\chi_{0m})$ and comparing the terms in (3.14), also by (3.1) one can easily calculate a constant $C_4$ (independent of $\delta, \varepsilon$, and $m$) satisfying

$$\|\chi_{0m}\|^2_{H^1(\Omega)} + \varepsilon\,\|\Delta\chi_{0m}\|^2 + \|\xi_{0m}\|^2 \le C_4 \tag{3.15}$$

for any $m \in \mathbb{N}$. Thanks to (3.15), there are two elements $\chi_{0\delta\varepsilon} \in H^2(\Omega)$, $\xi_{0\delta\varepsilon} \in L^2(\Omega)$ such that, possibly taking subsequences, $\chi_{0m} \to \chi_{0\delta\varepsilon}$ and $\xi_{0m} \to \xi_{0\delta\varepsilon}$ weakly in the respective spaces, as $m \nearrow \infty$. Hence, by compactness we have that

$$\chi_{0m} \to \chi_{0\delta\varepsilon} \quad \text{strongly in } L^2(\Omega) \tag{3.16}$$

and consequently $(\xi_{0m}, \chi_{0m}) \to (\xi_{0\delta\varepsilon}, \chi_{0\delta\varepsilon})$ as $m \nearrow \infty$. Therefore, recalling [3, Prop. 1.1, p. 42] and passing to the limit in (3.14), we infer that $\xi_{0\delta\varepsilon}, \chi_{0\delta\varepsilon}$ fulfil (3.4) and $\chi_{0\delta\varepsilon}$ actually gives the unique solution to (3.12), (2.34). The above convergences hold for the whole sequences and, on account of the weak semicontinuity of norms, the estimate (3.10) is still valid for $\chi_{0\delta\varepsilon}$ and $\xi_{0\delta\varepsilon}$. Then, letting $u_{0\delta\varepsilon} = u_0 + a(\chi_{0\delta\varepsilon} - \chi_0)/2$, in view of (2.7)–(2.8) and (3.1) it is straightforward to deduce (3.5)–(3.6) and to check that $\xi_{0\delta\varepsilon}$ just coincides with $\varepsilon\,\Delta\chi_{0\delta\varepsilon} + L(u_C - u_{0\delta\varepsilon})$. Now, (3.8) is trivially satisfied if $(\mathbf{P}_\varepsilon)$ is concerned (see (2.31), (2.48), (2.53)), else it follows from a passage to the limit in (2.55)–(2.56) as $\varepsilon \searrow 0$. Indeed, it turns out that (cf. (3.15), (3.5), and (2.29)) $\varepsilon\,\Delta\chi_{0\delta\varepsilon} \to 0$ strongly in $L^2(\Omega)$ and $u_{0\delta\varepsilon} \to u_0$, $\chi_{0\delta\varepsilon} \to \chi_0$ weakly in $H^1(\Omega)$ (and strongly in $L^2(\Omega)$), the limits $u_0$ and $\chi_0$ being uniquely found by reason of (2.55) and (2.29). Due to (3.5), (2.31)–(2.33), and (2.5), one recovers weak convergences of $\theta_{0\delta\varepsilon}$ to $\theta_0 = 1/u_0$, and thus of $e_{0\delta\varepsilon}$ to $e_0$, even in $H^1(\Omega)$. By achieving the proof of the lemma, let us point out that in all three cases we have obtained something more than (3.6) and (3.8) ($\|\chi_{0\delta\varepsilon}\|_{H^1(\Omega)}$ uniformly bounded and $e_{0\delta\varepsilon} \to e_0$ strongly in $L^2(\Omega)$), but the statement of the lemma expresses what we actually need in the further analysis and, at the same time, it yields the essential requirements for alternative regularizing sequences. $\square$

After discussing the properties of approximating data, we are going to treat the problem $(\mathbf{P}_{\delta\,\varepsilon})$ and prepare some inequalities fulfilled by $\chi_{\delta\,\varepsilon}$ and $u_{\delta\,\varepsilon}$. For the moment, we work (first) on (2.43) and (then) on (2.42) separately.

**Lemma 3.3.** *For any $\delta\,,\varepsilon$ obeying (3.1) the solution $(\theta_{\delta\,\varepsilon}\,, u_{\delta\,\varepsilon}\,, \chi_{\delta\,\varepsilon}\,, \xi_{\delta\,\varepsilon})$ of Problem $(\boldsymbol{P}_{\delta\,\varepsilon})$ satisfies*

$$\frac{\delta}{2}\,\|\partial_t\,\chi_{\delta\,\varepsilon}\,(\,\cdot\,,\,t)\|^2 \,+\, \varepsilon\,\int_0^t \|\nabla\,(\partial_t\,\chi_{\delta\,\varepsilon})\,(\,\cdot\,,\,\tau)\|^2\,d\,\tau$$

$$\leq\,\frac{C_3}{2}\,-\,L\,\int_0^t\int_\Omega(\partial_t\,u_{\delta\,\varepsilon})\,(\partial_t\,\chi_{\delta\,\varepsilon}) \quad \textit{for a.e. } t\,\in\,(0\,,T)\,, \tag{3.17}$$

$$\|\xi_{\delta\,\varepsilon}(\,\cdot\,,\,t)\|^2\,\leq\,2\,\|L\Big(u_C\,-\,u_{\delta\,\varepsilon}(\,\cdot\,,\,t)\Big)\|^2\,+\,2\,\delta\,C_3$$

$$-\,4\,L\,\delta\,\int_0^t\int_\Omega(\partial_t\,u_{\delta\,\varepsilon})\,(\partial_t\,\chi_{\delta\,\varepsilon}) \quad \textit{for a.e. } t\,\in\,(0\,,T)\,, \tag{3.18}$$

$$\frac{\delta}{2}\,\|\nabla\chi_{\delta\,\varepsilon}\,(\,\cdot\,,\,t)\|^2\,+\,\varepsilon\,\int_0^t\,\|\Delta\chi_{\delta\,\varepsilon}\,(\,\cdot\,,\,\tau)\|^2 d\,\tau$$

$$\leq\,\frac{C_2}{2}\,-\,L\,\int_0^t\int_\Omega\nabla u_{\delta\,\varepsilon}\,\cdot\,\nabla\chi_{\delta\,\varepsilon} \qquad \forall\;t\,\in\,[0\,,T]\,, \tag{3.19}$$

*where the constants $C_2\,,C_3$ are characterized in Lemma 3.2.*

**Proof.** In order to show (3.17)–(3.19) rigorously, we use again the Yosida regularization (3.13) of the graph $\beta$. Therefore, for $m\,\in\,\mathbb{N}$ let $\chi_m$ be the solution to the system

$$\delta\,\partial_t\,\chi_m\,-\,\varepsilon\,\Delta\chi_m\,+\,\beta_m(\chi_m)\,=\,L\,(u_C\,-\,u_{\delta\,\varepsilon}) \quad \text{a.e. in } Q\,, \tag{3.20}$$

$$\frac{\partial\,\chi_m}{\partial\,n}\,=\,0 \qquad \text{a.e. in } \Sigma\,, \tag{3.21}$$

$$\chi_m\,(\,\cdot\,,\,0)\,=\,\chi_{0\,m} \quad \text{a.e. in } \Omega\,, \tag{3.22}$$

with $\chi_{0\,m}$ fulfilling (2.34) and (3.14) under the setting of Theorems 2.9 and 2.10, or $\chi_{0\,m}\,=\,\chi_{0\,\delta\,\varepsilon}$ in the framework of Theorem 2.8. Owing to (3.20) and (3.13), $\chi_m$ is smoother than $\chi_{\delta\,\varepsilon}$. Namely, for any $m\,\in\,\mathbb{N}$ we have that (see, e.g., [12, Lemma 4.1])

$$\chi_m\,\in\,H^2(s\,,T;\,L^2(\Omega))\,\cap\,H^1(s\,,T;\,H^2(\Omega)) \quad \forall\;s\,\in\,(0\,,T)\,,$$

$$\chi_m\,\in\,C^1([0\,,T];\,L^2(\Omega))\,\cap\,C^0([0\,,T];\,H^2(\Omega))$$

besides (2.38). By exploiting the above regularity, we can get the a priori estimate leading to (3.17). In fact, we differentiate (3.20) with respect to time, multiply by $\partial_t\,\chi_m$, and integrate over $\Omega\,\times\,(s\,,t)$ for $0\,<\,s\,<\,t\,<\,T$. Since $\beta_m'\,\geq\,0$ a.e. in $\mathbb{R}$, thanks to (3.21) we infer that

$$\frac{\delta}{2}\,\|\partial_t\,\chi_m(\,\cdot\,,\,t)\|^2\,+\,\varepsilon\,\int_s^t\,\|\nabla\,(\partial_t\,\chi_m)\,(\,\cdot\,,\,\tau)\|^2 d\,\tau$$

$$\leq\,\frac{\delta}{2}\,\|\partial_t\,\chi_m\,(\,\cdot\,,\,s)\|^2\,-\,L\,\int_s^t\int_\Omega(\partial_t\,u_{\delta\,\varepsilon})\,(\partial_t\,\chi_m)\,, \tag{3.23}$$

and then take the limit in this inequality as $s \searrow 0$. Moreover, note that (cf. (3.20), (3.22), (2.45), (2.33)–(2.35), and (3.13))

$$\partial_t \chi_m(\cdot\,,0) = \frac{1}{\delta}\Big(\varepsilon\,\Delta\chi_{0\delta\varepsilon} + L\,(u_C - u_{0\delta\varepsilon})\Big)$$

if $\chi_{0m} = \chi_{0\delta\varepsilon}$ is given by (2.49), and that (cf. also (3.14) and (2.55))

$$\partial_t \chi_m(\cdot\,,0) = \frac{L\,a}{2\,\delta}(\chi_{0m} - \chi_{0\delta\varepsilon})$$

if $\chi_{0m}$ solves (2.34), (3.14). In any case, accounting for Lemma 3.2 it is not difficult to see that

$$\frac{\delta}{2}\|\partial_t \chi_m(\cdot\,,0)\|^2 \le \frac{C_3}{2} + \frac{L^2\,a^2}{8\,\delta}\|\chi_{0m} - \chi_{0\delta\varepsilon}\|^2, \tag{3.24}$$

where the last term goes to 0 as $m \nearrow \infty$ because of (3.16). Next, let us just outline the deduction of estimates like (3.18)–(3.19) for $\chi_m$ and $\xi_m = \beta_m(\chi_m)$. Concerning (3.18), it suffices to test (3.20) by $\xi_m$ and integrate only in space, using (3.21) and the positiveness of $\beta_m'$. Then one applies the Young inequality and the bound already found for $\|\partial_t \chi_m(\cdot\,,t)\|^2$ (i.e., (3.23) with $s = 0$ plus (3.24)). To obtain (3.19) we multiply (3.20) by $-\Delta\chi_m$ and integrate by parts in space and time. The constant in the right hand side is due to (3.6) (cf. the proof of Lemma 3.2). In conclusion, the earlier estimates and (2.37) ($u_{\delta\varepsilon}$ is fixed in such argument) enable us to pass to the limit in (3.20)–(3.22) as $m \nearrow \infty$ by compactness, and to establish that the weak star limits $\chi_{\delta\varepsilon}, \xi_{\delta\varepsilon}$ of the sequences $\chi_m, \xi_m$ satisfy (3.17)–(3.19) (this last part is more detailed in [6, Lemma 3.1]). ☐

**Lemma 3.4.** *There are two constants $C_5$ and $C_6$, independent of the parameters $\delta$ and $\varepsilon$ in (3.1), such that the solution $(\theta_{\delta\varepsilon}, u_{\delta\varepsilon}, \chi_{\delta\varepsilon}, \xi_{\delta\varepsilon})$ of Problem $(\boldsymbol{P}_{\delta\varepsilon})$ fulfils*

$$\begin{aligned}
&\frac{c_0}{2}\int_0^t\!\!\int_\Omega \Big|\frac{\partial_t u_{\delta\varepsilon}}{u_{\delta\varepsilon}}\Big|^2 + \frac{k}{2}\|\nabla u_{\delta\varepsilon}(\cdot\,,t)\|^2 + \frac{c}{4}\|u_{\delta\varepsilon}(\cdot\,,t)\|_\Gamma^2 \\
&\le C_5\Big(1 + \int_0^t \big(\|\nabla u_{\delta\varepsilon}(\cdot\,,\tau)\|^2 + \|u_{\delta\varepsilon}(\cdot\,,\tau)\|_\Gamma^2\big)d\tau\Big) \\
&\quad + L\int_0^t\!\!\int_\Omega (\partial_t \chi_{\delta\varepsilon})(\partial_t u_{\delta\varepsilon}) \quad \textit{for a.e. } t \in (0,T),
\end{aligned} \tag{3.25}$$

*the constant $c$ being introduced in (2.3), and*

$$\begin{aligned}
&c_0\int_0^t\!\!\int_\Omega \frac{|\nabla u_{\delta\varepsilon}|^2}{u_{\delta\varepsilon}^2} + \frac{k}{4}\Big\|\Delta\int_0^t u_\varepsilon(\cdot\,,\tau)d\tau\Big\|^2 \\
&\le C_6\Big(1 + \int_0^t \Big(\|u_{\delta\varepsilon}(\cdot\,,\tau)\|_\Gamma^2 + \|\Delta\int_0^\tau u_{\delta\varepsilon}(\cdot\,,s)ds\|^2\Big)d\tau\Big) \\
&\quad + L\int_0^t\!\!\int_\Omega \nabla\chi_{\delta\varepsilon}\cdot\nabla u_{\delta\varepsilon} \qquad \forall\ t \in [0,T].
\end{aligned} \tag{3.26}$$

**Proof.** A precise derivation of the inequality (3.25) needs some preliminary regularization of $(\mathbf{P}_{\delta\varepsilon})$ or, at least, of (2.42). Referring to [20] or [14] for this matter, let us proceed formally. Testing (2.42) by $-\partial_t u_{\delta\varepsilon}$, integrating, and applying formal Green formulas, with the help of (2.40), (2.44)–(2.45), (2.3)–(2.4), and (2.32)–(2.33) we get the identity

$$
\begin{aligned}
& c_0 \int_0^t \int_\Omega \Big| \frac{\partial_t u_{\delta\varepsilon}}{u_{\delta\varepsilon}} \Big|^2 + \frac{k}{2} \|\nabla u_{\delta\varepsilon}(\,\cdot\,, t)\|^2 + \frac{1}{2} \int_\Gamma (\gamma\, u_{\delta\varepsilon}^2)(\,\cdot\,, t) \\
&= \frac{k}{2} \|\nabla u_{0\delta\varepsilon}\|^2 + \int_\Gamma (\zeta\, u_{\delta\varepsilon})(\,\cdot\,, t) + \frac{1}{2} \int_\Gamma \Big( \gamma(\,\cdot\,, 0) u_{0\delta\varepsilon}^2 - 2\,\zeta(\,\cdot\,, 0) u_{0\delta\varepsilon} \Big) \\
&\quad + \frac{1}{2} \int_0^t \int_\Gamma (\gamma_t\, u_{\delta\varepsilon}^2 - 2\,\zeta_t\, u_{\delta\varepsilon}) - \int_0^t \int_\Omega g_{\delta\varepsilon}\, \partial_t u_{\delta\varepsilon} \\
&\quad + L \int_0^t \int_\Omega (\partial_t \chi_{\delta\varepsilon})\, (\partial_t u_{\delta\varepsilon}) \quad \text{for a.e. } t \in (0\,, T)\,.
\end{aligned}
\tag{3.27}
$$

Thanks to (2.3)–(2.4) and (3.2) we have that

$$
\begin{aligned}
\frac{1}{2} \int_\Gamma (\gamma\, u_{\delta\varepsilon}^2)(\,\cdot\,, t) &\geq \frac{c}{2} \|u_{\delta\varepsilon}(\,\cdot\,, t)\|_\Gamma^2\,, \\
\int_\Gamma (\zeta\, u_{\delta\varepsilon})(\,\cdot\,, t) &\leq \frac{1}{c} \|\zeta\|_{L^\infty(0,T;L^2(\Gamma))}^2 + \frac{c}{4} \|u_{\delta\varepsilon}(\,\cdot\,, t)\|_\Gamma^2\,, \\
- \int_0^t \int_\Omega g_{\delta\varepsilon}\, \partial_t u_{\delta\varepsilon} &\leq \|g_{\delta\varepsilon}\|_{L^\infty(Q)} \int_0^t \int_\Omega \Big| \frac{\partial_t u_{\delta\varepsilon}}{u_{\delta\varepsilon}} \Big| u_{\delta\varepsilon} \\
&\leq \frac{c_0}{2} \int_0^t \int_\Omega \Big| \frac{\partial_t u_{\delta\varepsilon}}{u_{\delta\varepsilon}} \Big|^2 + \frac{C_1^2}{2\, c_0} \int_0^t \|u_{\delta\varepsilon}(\,\cdot\,, \tau)\|^2 d\tau\,.
\end{aligned}
$$

Then, recalling also (3.5) and letting $\omega$ denote a constant such that

$$
\|v\|_{H^1(\Omega)}^2 \leq \omega \Big( \|\nabla v\|^2 + \|v\|_\Gamma^2 \Big) \qquad \forall\ v \in H^1(\Omega)\,,
\tag{3.28}
$$

from (3.27) it is straightforward to deduce (3.25), where $C_5$ depends only upon $k\,, c\,, c_0\,, \omega\,, C_1\,,$ $C_2\,, a\,, b\,, \|\gamma\|_{L^\infty(\Sigma)}\,, \|\gamma_t\|_{L^\infty(\Sigma)}\,, \|\zeta\|_{H^1(0,T;L^2(\Gamma))}\,,$ and on the bi–dimensional surface measure $\mathcal{H}^2(\Gamma)$ of the boundary $\Gamma$. At this point, it remains to show (3.26), which does not require any regularization of (2.42). In view of (2.36)–(2.37), (2.45), and (2.31), we remark that

$$
\begin{aligned}
& (c_0\, \theta_{\delta\varepsilon} + L\, \chi_{\delta\varepsilon})(\,\cdot\,, s) + k\, \Delta \int_0^s u_{\delta\varepsilon}(\,\cdot\,, \tau)\, d\tau = e_{0\delta\varepsilon} + \int_0^s g_{\delta\varepsilon}(\,\cdot\,, \tau)\, d\tau \\
&\qquad\qquad \text{a.e. in } \Omega\,, \qquad \forall\ s \in [0\,, T]\,.
\end{aligned}
\tag{3.29}
$$

Multiplying (3.29) by $\Delta u_{\delta\varepsilon}(\,\cdot\,, s)$ and integrating over $\Omega \times (0\,, t)$, from (2.40) and (2.44) it results that

$$
\begin{aligned}
& c_0 \int_0^t \int_\Omega \frac{|\nabla u_{\delta\varepsilon}|^2}{u_{\delta\varepsilon}^2} + \frac{k}{2} \Big\|\Delta \int_0^t u_{\delta\varepsilon}(\,\cdot\,, \tau)\, d\tau\Big\|^2 + \frac{c_0}{k} \int_0^t \int_\Gamma \zeta\, \theta_{\delta\varepsilon} \\
&= \frac{1}{k} \int_0^t \int_\Gamma \Big( c_0\, \gamma + L\, \chi_{\delta\varepsilon}\, (\gamma\, u_{\delta\varepsilon} - \zeta) \Big) \\
&\quad + \Big( e_{0\delta\varepsilon} + \int_0^t g_{\delta\varepsilon}(\,\cdot\,, \tau)\, d\tau\,, \Delta \int_0^t u_{\delta\varepsilon}(\,\cdot\,, \tau)\, d\tau \Big) \\
&\quad - \int_0^t \Big( g_{\delta\varepsilon}(\,\cdot\,, \tau)\,, \Delta \int_0^\tau u_{\delta\varepsilon}(\,\cdot\,, s)\, ds \Big)\, d\tau \\
&\quad + L \int_0^t \int_\Omega \nabla \chi_{\delta\varepsilon} \cdot \nabla u_{\delta\varepsilon} \qquad \forall\ t \in [0\,, T]\,.
\end{aligned}
$$

Hence, since (cf. (2.40)–(2.41), (2.3)–(2.4), and (2.31))

$$\frac{c_0}{k}\int_0^t\int_\Gamma \zeta\,\theta_{\delta\,\varepsilon} \,\geq\, 0\,,$$

$$\frac{1}{k}\int_0^t\int_\Gamma \Big(c_0\,\gamma \,+\, L\,\chi_{\delta\,\varepsilon}\,(\gamma\,u_{\delta\,\varepsilon} \,-\, \zeta)\Big) \,\leq\, \int_0^t\int_\Gamma \frac{\gamma}{k}\,(c_0 \,+\, L\,u_{\delta\,\varepsilon})\,,$$

$$\Big(e_{0\,\delta\,\varepsilon} \,+\, \int_0^t g_{\delta\,\varepsilon}\,(\,\cdot\,,\,\tau)\,d\tau\,,\,\Delta\int_0^t u_{\delta\,\varepsilon}\,(\,\cdot\,,\,\tau)\,d\tau\Big)$$

$$\leq\, \frac{1}{k}\Big\|e_{0\,\delta\,\varepsilon} \,+\, \int_0^t g_{\delta\,\varepsilon}\,(\,\cdot\,,\,\tau)\,d\tau\Big\|^2 \,+\, \frac{k}{4}\Big\|\Delta\int_0^t u_{\delta\,\varepsilon}\,(\,\cdot\,,\,\tau)\,d\tau\Big\|^2\,,$$

and $e_{0\,\delta\,\varepsilon} \,=\, c_0\,\theta_{0\,\delta\,\varepsilon} + L\,\chi_{0\,\delta\,\varepsilon}$, owing to (2.3), (2.33), (2.35), (3.2), and (3.5) it is not difficult to determine a constant $C_6$, depending just on $L\,,k\,,c_0\,,a\,,C_1\,,\mathcal{H}^2(\Gamma)\,,\|\gamma\|_{L^2(\Sigma)}\,,|\Omega|\,,$ and $T$, such that (3.26) holds. Therefore the lemma is completely proved. □

By combining the inequalities (3.17)–(3.19) with (3.25)–(3.26), we finally obtain global estimates for the sequences of functions $\theta_{\delta\,\varepsilon}\,,u_{\delta\,\varepsilon}\,,\chi_{\delta\,\varepsilon}\,,\xi_{\delta\,\varepsilon}$ considered in Theorems 2.8–2.10. From now on, let us omit specifications in the statements, being understood that $\delta$ and $\varepsilon$ satisfy (3.1).

**Lemma 3.5.** *There exists a constant $C_7$ such that*

$$\|\ln(u_{\delta\,\varepsilon})\|^2_{H^1(0,T;L^2(\Omega))} \,+\, \|u_{\delta\,\varepsilon}\|^2_{L^\infty(0,T;H^1(\Omega))} \,+\, \delta\,\|\chi_{\delta\,\varepsilon}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}$$

$$+\,\varepsilon\,\|\nabla\chi_\varepsilon\|^2_{H^1(0,T;(L^2(\Omega))^3)} \,+\, \|\chi_{\delta\,\varepsilon}\|_{L^\infty(Q)} \,+\, \|c_0\,\theta_{\delta\,\varepsilon} \,+\, L\,\chi_{\delta\,\varepsilon}\|_{W^{1,\infty}(0,T;V')} \,\leq\, C_7\,. \tag{3.30}$$

**Proof.** Take the sum of (3.17) and (3.25), then apply Gronwall's lemma. As $(\partial_t\,u_{\delta\varepsilon})/u_{\delta\varepsilon} \,=\, \partial_t\,\ln(u_{\delta\,\varepsilon})$ a.e. in $Q$ and $\|\ln(u_{0\,\delta\,\varepsilon})\|_{L^\infty(\Omega)}$ is uniformly bounded because of (3.5), to achieve (3.30) it suffices to recall (3.28), (3.6), (2.41) and to make use of (2.46) along with (2.3)–(2.4) and (3.2). □

**Lemma 3.6.** *There is a constant $C_8$ such that*

$$\|\xi_{\delta\,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} \,+\, \varepsilon\,\|\chi_\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \,\leq\, C_8\,. \tag{3.31}$$

**Proof.** Multiply (3.25) by $4\,\delta$ and add it to (3.18). Due to (3.30) and (3.1), the right hand side of the resulting inequality is bounded independently of $\delta$ and $\varepsilon$. Next, a comparison of the terms in (2.43) allows us to control $\|\varepsilon\,\Delta\chi_{\delta\,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))}$, whence (3.31) follows by virtue of the boundary condition in (2.44). □

**Lemma 3.7.** *There exists a constant $C_9$ such that*

$$\|\ln(u_{\delta\,\varepsilon})\|^2_{L^2(0,T;H^1(\Omega))} \,+\, \|\theta_{\delta\,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$+\,\delta\,\|\chi_{\delta\,\varepsilon}\|^2_{L^\infty(0,T;H^1(\Omega))} \,+\, \varepsilon\,\|\chi_{\delta\,\varepsilon}\|^2_{L^2(0,T;H^2(\Omega))} \,\leq\, C_9\,. \tag{3.32}$$

**Proof.** The sum of (3.19) and (3.26), the proof of (3.30)–(3.31), and the Gronwall lemma lead to (3.32). Indeed, the boundedness of

$$\sup_{0\le t\le T} \left\| \Delta \int_0^t u_{\delta\,\varepsilon}(\,\cdot\,,\,\tau)\,d\tau \right\|$$

implies the boundedness of

$$\sup_{0\le s\le T} \|\theta_{\delta\,\varepsilon}(\,\cdot\,,\,s)\|$$

via (3.29). $\square$

**Lemma 3.8.** *There is a constant* $C_{10}$ *such that*

$$\|u_{\delta\,\varepsilon}\|_{H^1(0,T;L^{3/2}(\Omega))} + \|\theta_{\delta\,\varepsilon}\|_{H^1(0,T;L^1(\Omega))} \le C_{10}\,. \tag{3.33}$$

**Proof.** Recall that in three space dimensions we have the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$. Since

$$\|\partial_t\, u_{\delta\,\varepsilon}\|^2_{L^2(0,T;L^{3/2}(\Omega))} = \int_0^T \left|\int_\Omega |u_{\delta\,\varepsilon}\,\partial_t\,\ln(u_{\delta\,\varepsilon})|^{3/2}\right|^{4/3}$$

$$\le \int_0^T \left|\int_\Omega u^6_{\delta\,\varepsilon}\right|^{1/3} \int_\Omega |\partial_t\,\ln(u_{\delta\,\varepsilon})|^2$$

$$\le \|u_{\delta\,\varepsilon}\|^2_{L^\infty(0,T;L^6(\Omega))}\,\|\partial_t\,\ln(u_{\delta\,\varepsilon})\|^2_{L^2(Q)}\,,$$

$\ln(\theta_{\delta\,\varepsilon}) = -\ln(u_{\delta\,\varepsilon})$ a.e. in $Q$ (see (2.40)), and

$$\|\partial_t\,\theta_{\delta\,\varepsilon}\|^2_{L^2(0,T;L^1(\Omega))} \le \|\theta_{\delta\,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega))}\,\|\partial_t\,\ln(u_{\delta\,\varepsilon})\|^2_{L^2(Q)}\,,$$

(3.33) is a straightforward consequence of (3.30) and (3.32). $\square$

**Lemma 3.9.** *There is a constant* $C_{11}$ *such that*

$$\varepsilon\,\|\chi_{\delta\,\varepsilon}\|^2_{H^1(0,T;H^1(\Omega))} \le C_{11}\,. \tag{3.34}$$

**Proof.** Picking $v = 1$ in (2.46), by (2.3)–(2.4) and (3.2) we realize that

$$L\left|\int_\Omega \partial_t\,\chi_{\delta\,\varepsilon}(\,\cdot\,,\,t)\right| \le c_0\,\|\partial_t\,\theta_{\delta\,\varepsilon}(\,\cdot\,,\,t)\|_{L^1(\Omega)}$$

$$+\,\|\gamma\|_{L^\infty(0,T;L^2(\Gamma))}\;\|u_{\delta\,\varepsilon}(\,\cdot\,,\,t)\|_\Gamma + \|\zeta\|_{L^\infty(0,T;L^1(\Gamma))} + C_1\,|\Omega|\,.$$

Then (3.34) results from (3.30), (3.33), and (3.1), using Poincaré's inequality. $\square$

## 4. Passage to the limit and existence

This section is devoted to pursue the proof of Theorems 2.8–2.10 and together prove the existence of solutions to the problems $(\mathbf{P}_\delta)$, $(\mathbf{P}_\varepsilon)$, $(\mathbf{P})$. Moreover, we make some comments about possible extensions of the results in several directions.

Thanks to the estimates (3.30)–(3.33), within all three frameworks there exist functions $\theta, u, \xi, \chi$ such that, in principle for subsequences, the convergences (2.50)–(2.51), (2.57), and

$$c_0\,\theta_{\delta\,\varepsilon} + L\,\chi_{\delta\,\varepsilon} \to c_0\,\theta + L\,\chi \qquad \text{weakly star in } W^{1,\infty}(0,T;\,V') \tag{4.1}$$

hold as $\varepsilon$, or $\delta$, or both $\varepsilon$ and $\delta$, go to $0$. From (2.51) and the Aubin compactness lemma (see, e.g., [16, p. 58]) we also get

$$u_{\delta\,\varepsilon} \to u \quad \text{strongly in } L^2(0,T;\,H^{1-r}(\Omega)) \qquad \text{for any } r > 0, \tag{4.2}$$

which implies (if $r < 1/2$) that

$$u_{\delta\,\varepsilon}{}_{|\Gamma} \to u_{|\Gamma} \quad \text{strongly in } L^2(\Sigma). \tag{4.3}$$

Therefore, recalling (3.3), (2.46), and (3.8), it is easy to verify that $\theta, u, \xi, \chi$ fulfil (2.9)–(2.10), (2.24), and (2.13)–(2.15). The condition (2.11) follows from (2.40), (2.50), (4.2), entailing that

$$1 = \theta_{\delta\,\varepsilon}\, u_{\delta\,\varepsilon} \to \theta\, u \quad \text{weakly in } L^1(Q).$$

The strong convergence of $u_{\delta\,\varepsilon}$ plays a role here, as well as in the next derivation of (2.12), at least for Problem $(\mathbf{P})$. Owing to (2.41) and (2.1), to show (2.12) it is enough to check that (cf., e.g., [3, Lemma 1.3, p. 42])

$$\limsup \int_0^T\!\!\int_\Omega \xi_{\delta\,\varepsilon}\,\chi_{\delta\,\varepsilon} \le \int_0^T\!\!\int_\Omega \xi\,\chi\,. \tag{4.4}$$

Let us now examine the three different cases of passage to the limit separately.

**Lemma 4.1.** *Under the assumptions of Theorem 2.8, let $\theta, u, \chi, \xi$ be weak star limits of $\theta_{\delta\,\varepsilon}, u_{\delta\,\varepsilon}, \chi_{\delta\,\varepsilon}, \xi_{\delta\,\varepsilon}$ for some subsequence of $\varepsilon \searrow 0$. Then $(\theta, u, \chi, \xi)$ yields a solution to Problem $(\boldsymbol{P}_\delta)$.*

**Proof.** Since $\delta$ is fixed and $\varepsilon$ tends to $0$, (3.30) and (3.32) give (2.52) and

$$\varepsilon\,\chi_{\delta\,\varepsilon} \to 0 \quad \text{strongly in } L^2(0,T;\,H^2(\Omega)), \tag{4.5}$$

to join with (2.50)–(2.51). Hence, accounting for (2.43), (2.45), and (3.9), it turns out that (2.16)–(2.18) and (2.25) are satisfied. From (2.52), by the Ascoli theorem, we infer that $\chi_{\delta\,\varepsilon} \to \chi$ strongly in $C^0([0,T];\,L^2(\Omega))$, which plainly ensures (4.4). But (4.4) can be recovered without using the last property, just exploiting the weak lower semicontinuity of norms. Indeed, observe that (see (2.43)–(2.45))

$$\begin{aligned}\int_0^T\!\!\int_\Omega \xi_{\delta\,\varepsilon}\,\chi_{\delta\,\varepsilon} &= L\int_0^T\!\!\int_\Omega (u_C - u_{\delta\,\varepsilon})\,\chi_{\delta\,\varepsilon} + \frac{\delta}{2}\,\|\chi_{0\,\delta\,\varepsilon}\|^2 \\ &\quad - \frac{\delta}{2}\,\|\chi_{\delta\,\varepsilon}(\,\cdot\,,T)\|^2 - \varepsilon\int_0^T\!\!\int_\Omega |\nabla\chi_{\delta\,\varepsilon}|^2,\end{aligned}$$

and consequently

$$\limsup_{\varepsilon \searrow 0} \int_0^T \int_\Omega \xi_{\delta\varepsilon}\, \chi_{\delta\varepsilon} \le L \int_0^T \int_\Omega (u_C - u)\chi + \frac{\delta}{2}\|\chi_0\|^2 - \frac{\delta}{2}\|\chi(\,\cdot\,, T)\|^2$$

because of (4.2), (2.52), and (3.9). Thus, due to (2.17)–(2.18) we easily obtain (4.4).

**Lemma 4.2.** *Under the assumptions of Theorem 2.9, let $\theta\,, u\,, \chi\,, \xi$ be weak star limits of $\theta_{\delta\varepsilon}\,, u_{\delta\varepsilon}\,, \chi_{\delta\varepsilon}\,, \xi_{\delta\varepsilon}$ for some subsequence of $\delta \searrow 0$. Then $(\theta\,, u\,, \chi\,, \xi\,)$ yields a solution to Problem $(\boldsymbol{P}_\varepsilon)$.*

**Proof.** Now besides (2.50), (2.51), (2.57) we have (cf. (3.30)–(3.31) and (3.34))

$$\delta\,\chi_{\delta\varepsilon} \to 0 \quad \text{strongly in } W^{1,\infty}(0\,,T;\, L^2(\Omega))\,, \tag{4.6}$$

and (2.54) as $\delta \searrow 0$, so that (2.20) and (2.21) result from (2.43) and (2.44). By (2.54) it is straightforward to deduce a strong convergence for $\chi_{\delta\varepsilon}$, whence (4.4) is certainly fulfilled. □

**Lemma 4.3.** *Under the assumptions of Theorem 2.10, let $\theta\,, u\,, \chi\,, \xi$ be weak star limits of $\theta_{\delta\varepsilon}\,, u_{\delta\varepsilon}\,, \chi_{\delta\varepsilon}\,, \xi_{\delta\varepsilon}$ for some subsequence of $\varepsilon \searrow 0$ and $\delta \searrow 0$. Then $(\theta\,, u\,, \chi\,, \xi\,)$ yields a solution to Problem $(\boldsymbol{P})$.*

**Proof.** In this case both (4.5) and (4.6) hold in addition to (2.50)–(2.51) and (2.57). Therefore, taking the limit in (2.43) as $\varepsilon$ and $\delta$ tend to $0$, we find (2.22). Moreover, since

$$\xi_{\delta\varepsilon} = L(u_C - u_{\delta\varepsilon}) - \delta\,\partial_t\,\chi_{\delta\varepsilon} + \varepsilon\,\Delta\chi_{\delta\varepsilon} \to L(u_C - u) = \xi \quad \text{strongly in } L^2(Q)\,,$$

by virtue of (4.2), we get readily (4.4). □

Having proved Lemmas 4.1–4.3, at the present level it remains to show that the three problems $(\mathbf{P}_\delta)$, $(\mathbf{P}_\varepsilon)$, $(\mathbf{P})$ possess only one solution, so to achieve the proof of Theorems 2.2–2.4 and 2.8–2.10. The uniqueness being accomplished in the next section under very few hypotheses on the data, let us discuss here some questions related to the results already set out.

**Remark 4.4.** Concerning Problem $(\mathbf{P}_\delta)$, the estimates (3.30)–(3.31), (3.33)–(3.34) can be replaced by the weaker ones

$$\|\theta_{\delta\varepsilon}\|_{L^\infty(0,T;L^1(\Omega))} + \|u_{\delta\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} + \delta\|\chi_{\delta\varepsilon}\|^2_{H^1(0,T;L^2(\Omega))}$$

$$+\,\varepsilon\|\chi_{\delta\varepsilon}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\chi_{\delta\varepsilon}\|_{L^\infty(Q)} + \|c_0\,\theta_{\delta\varepsilon} + L\,\chi_{\delta\varepsilon}\|_{H^1(0,T;V')} \le C_{12}\,,$$

$$\|\xi_{\delta\varepsilon}\|_{L^2(Q)} \le C_{13}\,,$$

where the former is obtained testing (2.42) by $-u_{\delta\varepsilon} + 1$ and (2.43) by $\partial_t\,\chi_{\delta\varepsilon}$, integrating, adding, etc., and the latter comes, for instance, from (3.32) and (2.43). Then it is however possible to pass to the limit by compactness, on the basis of the strong convergences $\chi_{\delta\varepsilon} \to$

$\chi$ in $C^0([0,T];\, L^2(\Omega))$ and $c_0\,\theta_{\delta\,\varepsilon} + L\,\chi_{\delta\,\varepsilon} \to c_0\,\theta + L\,\chi$ in $C^0([0,T];\, V')$ as $\varepsilon \searrow 0$. In fact, note that

$$\theta_{\delta\,\varepsilon}\, u_{\delta\,\varepsilon} \;=\; \frac{1}{c_0}\,(c_0\,\theta_{\delta\,\varepsilon} \,+\, L\,\chi_{\delta\,\varepsilon})u_{\delta\,\varepsilon} \;-\; \frac{L}{c_0}\,\chi_{\delta\,\varepsilon}\, u_{\delta\,\varepsilon} \quad \text{a.e. in } Q\,,$$

and $u_{\delta\,\varepsilon} \to u$ weakly in $L^2(0,T;\, V)$. Further, (4.3) is not needed to take the limit in (2.46) (the boundary integral is linear with respect to $u_{\delta\,\varepsilon|_\Gamma}$). Obviously, this approach leads to a solution not so regular as in Theorem 2.2, but it permits to weaken the assumptions on $g\,,\gamma\,,\zeta$ and $u_0$ (the details of the alternative formulation are left to the reader).

**Remark 4.5.** It is addressed still to Problem $(\mathbf{P}_\delta)$. The condition $\chi_0 \in H^1(\Omega)$ is not necessary to achieve Theorem 2.2. Actually, letting $\chi_0 \in L^\infty(\Omega)$ lie between $0$ and $1$, we can reach the same conclusion (without (2.25)) by avoiding the estimate (3.19) (what happens is that (3.6) is no longer true with respect to $\delta$). Thus, after the deduction of (3.30)–(3.31) one simply chooses $v = \theta_{\delta\,\varepsilon}$ in (2.46), integrates in time, and uses the uniform boundedness of $\|\partial_t\,\chi_{\delta\,\varepsilon}\|_{L^2(Q)}$ stated in (3.30). The validity of (3.32) is then restricted to the first two terms and

$$\chi_{\delta\,\varepsilon} \to \chi \quad \text{weakly star in } W^{1,\infty}(0\,,T;\, L^2(\Omega)) \tag{4.7}$$

instead of (2.52), although (2.50)–(2.51) and (4.7) are sufficient to identify the limit problem (cf. the proof of Lemma 4.1). Indeed, a strong convergence for $\chi_{\delta\,\varepsilon}$ can be inferred from (2.43) and (4.2) by a direct argument. This is precisely done in the paper [6] (see Lemma 4.1 therein), where the nonlinearities $\lambda$ and $\sigma$ of (1.5)–(1.6) are included in $(\mathbf{P}_\delta)$ (and the strong convergence of $\chi_{\delta\,\varepsilon}$ becomes very important).

**Remark 4.6.** Regarding Problem $(\mathbf{P}_\varepsilon)$ and the regularity of its solution (see also Remark 2.5), the claim is that $\theta\,,u$ fulfil (2.36)–(2.37) as in the extended problem $(\mathbf{P}_{\delta\,\varepsilon})$. Indeed, the point is proving that $\theta\,,u \in L^\infty(Q)$ and, since (cf. (2.28)) $\chi_t \in L^2(0\,,T;\, L^6(\Omega))$, Lemmas 2.3–2.4 of [13] should fit with minor changes. The technique, already employed in [20], is based on Moser iteration procedures. A more delicate question is about the possibility of generalizing Theorem 2.3 and Theorem 2.9 to the Penrose–Fife system (1.5)–(1.6) with $\delta = 0$. Apparently, the analysis of Section 3 (cf. especially Lemma 3.3) works only if $\sigma'$ and $\lambda'$ are strictly decreasing functions, while, at least for $\sigma'$, we do not expect (cf., e.g., [18, 19]) monotonicity properties.

**Remark 4.7.** Let us come back to the issue raised in Remark 2.11. We would like to discard the assumption $\chi_0 \in H^1(\Omega)$ in Problem $(\mathbf{P})$. On the other hand, the approximating sequences of initial data must satisfy (3.5)–(3.8) in order to find solutions of $(\mathbf{P})$ by our asymptotics. For example, in the case when $u_0 \neq u_C$ a.e. in $\Omega$ (thus admitting sharp initial interfaces) we can give a positive answer and construct sequences $u_{0\,\delta\,\varepsilon}\,,\chi_{0\,\delta\,\varepsilon}$ complying with (3.5)–(3.8). In such a framework, since the condition (2.29) uniquely determines $\chi_0$, we can take $u_{0\,\delta\,\varepsilon} = u_0$ and $\chi_{0\,\delta\,\varepsilon}$ solving (2.34) and

$$\sqrt{\delta}\,\chi_{0\,\delta\,\varepsilon} \,-\, \varepsilon\,\Delta\chi_{0\,\delta\,\varepsilon} \,+\, \beta(\chi_{0\,\delta\,\varepsilon}) \ni L(u_C \,-\, u_0) \quad \text{a.e. in } \Omega \tag{4.8}$$

for all $\delta > 0$, $\varepsilon > 0$. It is a choice different from (2.55)–(2.56), though (2.35), (3.4), (3.5), and (3.7) still hold with $\xi_{0\,\delta\,\varepsilon} \;=\; -\sqrt{\delta}\,\chi_{0\,\delta\,\varepsilon} \,+\, \varepsilon\,\Delta\chi_{0\,\delta\,\varepsilon} \,+\, L(u_C \,-\, u_0)$. To verify

(3.6) it suffices to test (a regularized version of) (4.8) by $\chi_{0\delta\varepsilon}$ and by $-\sqrt{\delta}\,\Delta\chi_{0\delta\varepsilon}$, here integrating by parts and exploiting (3.5) and the Young inequality. Passing to the limit in (4.8) as $\varepsilon \searrow 0$ and $\delta \searrow 0$, and arguing like in (4.4), we easily recover (2.29) and then (cf. (2.31), (2.33), (2.5), (2.8)) also (3.8) is fulfilled. Therefore, the existence and convergence results in Theorems 2.4 and 2.10 remain valid even if $\chi_0 \notin H^1(\Omega)$, provided that the (three–dimensional) Lebesgue measure of the set $\{x \in \Omega : u_0(x) = u_C\}$ is zero.

**Remark 4.8.** It is a general remark concerning alternative boundary conditions to couple with (1.7). Referring to [6, Section 5], where the various approaches of [9–15, 20, 22] are discussed, one could wonder whether Theorems 2.2–2.4 and 2.8–2.10 extend to boundary conditions of the form

$$-k\frac{\partial u}{\partial n} = \gamma u^p - \zeta u^q \quad \text{in } \Sigma \tag{4.9}$$

with $p \geq 1, q > 0, p > q$. We do not know anything about uniqueness and, in this case, Proposition 2.6 only states the existence of a smooth solution to Problem $(\mathbf{P}_{\delta\varepsilon})$ for any $\delta > 0$ and any $\varepsilon > 0$. But the convergences in Theorems 2.8–2.10 turn out for subsequences of such solutions, thus assuring that there exist solutions of $(\mathbf{P}_\delta)$, $(\mathbf{P}_\varepsilon)$, $(\mathbf{P})$ even when $p$ and $q$ are different from $1$ and $0$, respectively. To justify our assertion, let us point out the few modifications in the proofs. By handling the estimate (3.25), Lemma 3.5 yields

$$\|u_{\delta\varepsilon|\Gamma}\|_{L^\infty(0,T;L^{p+1}(\Gamma))} \leq C_{14} \tag{4.10}$$

in addition to (3.30), so that one can easily control the actual right hand side of (3.26) to get (3.32). Moreover, (3.34) still follows. In view of (4.10) and (4.3), we deduce that, at least for a subsequence,

$$u_{\delta\varepsilon|\Gamma} \to u_{|\Gamma} \text{ weakly in } L^{p+1}(\Sigma) \text{ and a.e. in } \Sigma\,.$$

Hence, with the help of the Egorov theorem it is not difficult to conclude that $u_{\delta\varepsilon|\Gamma} \to u_{|\Gamma}$ strongly in $L^p(\Sigma)$, which enables us to pass to the limit in the variational equality corresponding to (2.46). Note that now the space $V$ of test functions must be restricted in order that the boundary term have a meaning. For instance, we can choose $V = H^2(\Omega)$ and consequently $v_{|\Gamma} \in L^\infty(\Gamma)$ for $v \in V$.

## 5. Uniqueness

Finally, we show the uniqueness properties stated in Theorems 2.2–2.4.

**Lemma 5.1** *Under the assumptions (2.1) and (2.3), each one of the three problems $(\boldsymbol{P}_\delta)$, $(\boldsymbol{P}_\varepsilon)$, $(\boldsymbol{P})$ admits at most one solution.*

**Proof.** Letting $\varepsilon$ and $\delta$ be zero or not, according to the cases, we try to unify the matter. Suppose that there are two solutions $(\theta_1, u_1, \chi_1, \xi_1)$ and $(\theta_2, u_2, \chi_2, \xi_2)$. Setting $\theta = \theta_1 - \theta_2$, $u = u_1 - u_2$, $\chi = \chi_1 - \chi_2$, $\xi = \xi_1 - \xi_2$ and integrating the two equations (2.14) with respect to time, we realize that (see (2.15) and (2.17) or (2.20) or (2.22))

$$c_0\Big(\theta(\,\cdot\,, s), v\Big) + L\Big(\chi(\,\cdot\,, s), v\Big) = k\int_\Omega \nabla \int_0^s u(\,\cdot\,, \tau)d\tau \cdot \nabla v$$

$$+\int_\Gamma \int_0^s (\gamma u)(\,\cdot\,,\tau)d\tau\, v \qquad \forall\, v \in V, \quad \text{for a.e. } s \in (0,T), \tag{5.1}$$

$$\delta \chi_t - \varepsilon \Delta\chi + \xi = -L u \qquad \text{a.e. in } Q. \tag{5.2}$$

Since (2.11) and (2.12) hold for both $\theta_i, u_i, \chi_i, \xi_i, i = 1, 2$, observe that

$$-\theta u = \frac{|u|^2}{u_1 u_2} \geq \frac{|u|^2}{1 + |u_1 u_2|} \quad \text{a.e. in } Q, \tag{5.3}$$

as well as $\xi\chi \geq 0$ because of the monotonicity of $\beta$. Then, multiplying (5.2) by $\chi$, and possibly using (2.18) or (2.21), the integration gives

$$\frac{\delta}{2}\|\chi(\,\cdot\,,t)\|^2 + \varepsilon \int_0^t \|\nabla\chi(\,\cdot\,,\tau)\|^2 d\tau \leq -L\int_0^t\int_\Omega u\chi \quad \forall\, t \in [0,T]. \tag{5.4}$$

On the other hand, taking $v = -u(\,\cdot\,,s)$ in (5.1) and integrating over $\Omega \times (0,t)$, owing to (5.3) and (2.3) we obtain

$$c_0 \int_0^t\int_\Omega \frac{|u|^2}{1+|u_1 u_2|} + \frac{k}{2}\|\nabla\int_0^t u(\,\cdot\,,\tau)d\tau\|^2$$

$$+\int_\Gamma \frac{1}{2\gamma(\,\cdot\,,t)}\left|\int_0^t (\gamma u)(\,\cdot\,,\tau)d\tau\right|^2 \leq -\int_0^t\int_\Gamma \frac{\gamma_t}{2\gamma^2}(\,\cdot\,,\tau)\left|\int_0^\tau(\gamma u)(\,\cdot\,,s)ds\right|^2 d\tau$$

$$+L\int_0^t\int_\Omega \chi u \qquad \forall\, t \in [0,T]. \tag{5.5}$$

Adding (5.5) to (5.4), noting that

$$-\frac{\gamma_t}{2\gamma^2} \leq \left\|\frac{\gamma_t}{\gamma}\right\|_{L^\infty(\Sigma)} \frac{1}{2\gamma} \quad \text{a.e. in } \Sigma,$$

and applying the Gronwall lemma, we infer that the sum of the left hand sides (of (5.4)–(5.5)) is equal to $0$ for any $t \in [0,T]$. Therefore it follows that $u = 0$ a.e. in $Q$, whence $u_1 = u_2$ and $\theta_1 = 1/u_1 = 1/u_2 = \theta_2$. At this point, (5.1) implies $\chi = 0$ so that $\xi = 0$ by (5.2), and the lemma is completely proved. $\square$

**Remark 5.2.** A global revision of the proof of Theorems 2.2–2.4 and 2.8–2.10 allows us to decide that the assumption (2.4) can be weakened. Actually, the requirement

$$\zeta \in H^1(0,T;\, L^2(\Gamma)), \quad \zeta \geq 0 \quad \text{a.e. in } \Sigma \tag{5.6}$$

serves our purposes (cf. especially Lemma 3.4). However, assuming (5.6) involves some regularization of $\zeta$ within Problem $(\mathbf{P}_{\delta\varepsilon})$, in order to exploit Proposition 2.6.

**Remark 5.3.** Thanks to Lemma 5.1, the convergences (2.50)–(2.52), (2.54), and (2.57) regard the whole sequences. On account of the convergence results, it would be interesting to investigate possible error estimates between the solutions to $(\mathbf{P}_{\delta\varepsilon})$ and to the limit problem. One method could be that developed in (5.1)–(5.5), but the expected outcome seems quite unsatisfactory. Then we let the question open.

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